

# CANONICAL REPRESENTATIONS FOR SINGLE-INPUT SINGLE-OUTPUT LINEAR TIME-VARYING SYSTEMS

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**Abstract**—*Scalar linear time-varying systems are addressed. In particular, canonical representations with integrators, multipliers and adders are presented. Essentially, it is shown that most of the well-known topologies for constant systems can be generalized to the time-varying context by replacing the conventional eigenvalues by the earlier introduced dynamic eigenvalues. However, one known topology can not be generalized. In stead, the adjoint system results.*

## I. INTRODUCTION

As is also shown in the accompanying paper, linear time-varying (LTV) systems arise as variational models of nonlinear dynamic systems [1]. Moreover, LTV systems model a variety of physical and technical systems with time-dependent parameters [2].

On the other hand, in classical filter design the solution of linear time-invariant (LTI) systems can be obtained by using integrators, multipliers and adders, respectively. Also, for such LTI signal processors different, but equivalent representations are known ([3], [4]).

In this article, canonical representations of linear *time-varying* (LTV) systems are derived. To be more precise, possible generalizations of known LTI-topologies to the LTV-context are studied. Then, it turns out that one known LTI-topology is not suitable for such a generalization. It is also shown that two other LTI-topologies, viz. a so-called series representation and the so-called cascade representation can be generalized indeed. In the latter, the conventional algebraic eigenvalues have to be replaced by the earlier introduced *dynamic* eigenvalues ([5], [6], [7], [8]).

Related results can be found in [9] and ([10], [11]). Essentially, these contributions are based on well-known mathematical methods for the factorization of the associated polynomial differential system operator (see also [12]). In contrast, our approach uses the Riccati transformation as described in [13]. Ba-

sically, this transformation effectuates an appropriate order reduction and a subsequent decoupling of the associated LTV system equations. A successive application of the above mentioned Riccati transformation *triangularizes* the accompanying LTV system matrix step-by-step ([7], [8], [14], [15]).

In the next section, first the original scalar differential equation is rewritten in a state-space system description. Then, an associated direct signal processor representation is easily obtained.

In Section 3, first an earlier obtained alternative method for the classical Cauchy-Floquet decomposition is shortly summarized [16]. Then, the cascade representation as described in [17] follows immediately. In this representation, the *dynamic* eigenvalues play the role of time-varying multipliers. As another result, it is shown that any triangularization step produces a next representation with one extra dynamic eigenvalue as a multiplier of the LTV processor.

Finally, in Section 4 it is shown that an alternative direct realization for LTI systems does not have a LTV antipode. In stead, the adjoint system is realized.

## II. A DIRECT SERIES REPRESENTATION

Consider the inhomogeneous scalar linear differential equation for the unknown  $x = x(t)$  with normalized time-varying coefficients  $a_i = a_i(t)$  ( $i = 1, \dots, n$ )

$$D^n x + a_n D^{n-1} x + \dots + a_2 D x + a_1 x = f, \quad (1)$$

in which  $D = d/dt$  while  $f = f(t)$  is a known function of time  $t$ . By introducing the new variables  $\{x_1, x_2, \dots, x_n\}$  as

$$x_1 = x, x_2 = \dot{x}_1, x_3 = \dot{x}_2, \dots, x_n = \dot{x}_{n-1}, \quad (2)$$

where the dot stands for a time-derivative, we obtain from (1)

$$\dot{x}_n = -a_n x_1 - \dots - a_1 x_n + f. \quad (3)$$

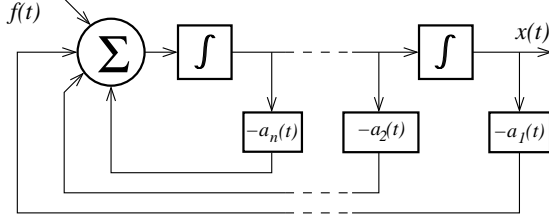


Fig. 1. A direct series representation.

On account of (2) and (3), the canonical direct series representation of figure 1 is easily recognized.

Next, introduce the  $n$ -dimensional column-vector  $\mathbf{x}$  with  $\mathbf{x}^T = [x_1, \dots, x_n]$  ( $T$  stands for the transpose) and unit vectors  $\mathbf{e}_j^{(n)}$ , ( $1 \leq j \leq n$ ), with  $[\mathbf{e}_j^{(n)}]^T = [\delta_{1j}, \dots, \delta_{nj}]$ , where  $\delta_{ij}$  denotes the Kronecker symbol. Then, the partitioned  $n \times n$  Frobenius companion system matrix  $\mathbf{A}(t)$  is introduced as

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{I}_{n-1}^+ & \mathbf{e}_{n-1}^{(n-1)} \\ -\mathbf{a}^T(t) & -a_1(t) \end{bmatrix}, \quad (4)$$

where  $\mathbf{I}_k^+$  denotes the  $k$ -dimensional square shift matrix, given by

$$\mathbf{I}_k^+ = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \dots & \dots & 0 \end{bmatrix}, \quad (5)$$

while the varying system parameters  $\{a_2, a_3, \dots, a_n\}$  are collected in the time-dependent row vector  $\mathbf{a}^T = [a_n, \dots, a_2]$ .

Now, the state-space description of (1) follows as

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{e}_n^{(n)} f(t), \quad (6)$$

with additional output equation

$$x = x_1. \quad (7)$$

As a result, equations (6) and (7) are represented by the LTV signal processor in figure 1.

### III. THE CASCADE AND HYBRID REPRESENTATION

As we showed earlier [8], there exists a lower triangular Riccati transformation matrix  $\mathbf{R}$

$$\mathbf{x}(t) = \mathbf{R}(t)\mathbf{y}(t), \quad (8)$$

by which system (6) is transformed into the new system

$$\dot{\mathbf{y}} = \{\text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) + \mathbf{I}_n^+\} \mathbf{y} + \mathbf{e}_n^{(n)} f \quad (9)$$

with  $\mathbf{y}^T = [y_1, \dots, y_n]$ , while the output equation (7) goes into

$$x = y_1. \quad (10)$$

Now, it is easily observed that the original differential equation (1) is given by the Cauchy-Floquet decomposition [16]

$$(D - \lambda_n(t))(D - \lambda_{n-1}(t)) \dots (D - \lambda_1(t))x = f(t). \quad (11)$$

Secondly, this decomposition constitutes the canonical cascade signal processor as depicted in figure 2. Note, that its multipliers  $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$  may be complex valued functions of the time  $t$ .

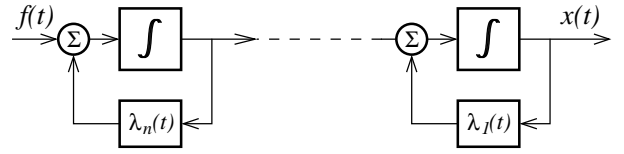


Fig. 2. The cascade topology.

As we explained earlier ([7], [8], [14]), each triangularization step needs a *particular* solution of a vector Riccati differential equation. If  $p_1, p_2, \dots$ , and  $p_{n-1}$  denotes the components of the solution vector of the first Riccati-equation, it is easily shown that the topology depicted in figure 3 is equivalent to the LTV signal processor of figure 1.

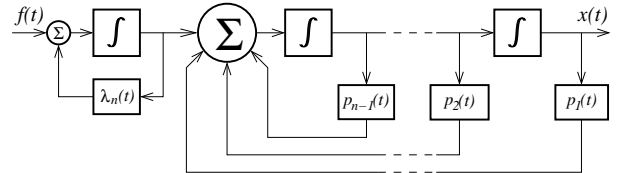


Fig. 3. A hybrid topology after one Riccati transformation.

The system matrix of system (9) indeed confirms that the functions  $\lambda_i(t)$  are a kind of eigenvalues. To show this rigorously, consider the homogeneous equation of system (9), thus  $f \equiv 0$ . This equation is investigated for modal solutions of the form

$$\mathbf{y}_j(t) = \mathbf{u}_j(t) \exp[\gamma_j(t)] \quad (12)$$

in which

$$\gamma_j(t) = \int_0^t \lambda_j(\tau) d\tau \quad (13)$$

and

$$\mathbf{u}_j(t) = [u_{1,j}(t), \dots, u_{j-1,j}(t), 1, 0, \dots, 0]^T. \quad (14)$$

Substitution of (12) and (14) in (9) with  $f \equiv 0$ , yields that (12) is indeed a solution, only if  $\mathbf{u}_j$  satisfies

$$\dot{\mathbf{u}}_j = [\{\text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) + \mathbf{I}_n^+\} - \lambda_j \mathbf{I}_n] \mathbf{u}_j, \quad (15)$$

where  $\mathbf{I}_n$  is the  $n$ -dimensional unity matrix. For a linear time-invariant system,  $\mathbf{u}_i(t)$  and  $\lambda_i(t)$  are constants and, as a consequence, the left hand side of (15) becomes zero. Hence, the classical eigenvalue problem results. As we argued earlier, this justifies to call  $\lambda_i(t)$  a *dynamic* eigenvalue and  $\mathbf{u}_i(t)$  a *dynamic* eigenvector ([5], [6], [7], [8]).

Now, it is clear that

$$\mathbf{U}(t) = [ \mathbf{u}_1(t), \dots, \mathbf{u}_n(t) ] \quad (16)$$

is a transformation matrix that transforms with

$$\mathbf{y}(t) = \mathbf{U}(t)\mathbf{z}(t) \quad (17)$$

the homogeneous equation (10) into

$$\dot{\mathbf{z}} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)\mathbf{z}. \quad (18)$$

Hence, the fundamental matrix  $\Phi = \Phi(t)$  of (1) is obtained as

$$\Phi = \mathbf{R}\mathbf{U} \text{diag}\{\exp(\gamma_1), \exp(\gamma_2), \dots, \exp(\gamma_n)\}, \quad (19)$$

in which  $\gamma_j = \gamma_j(t)$  is given by (13).

#### IV. AN ALTERNATIVE CONFIGURATION

Finally, consider the signal processor topology of figure 4.

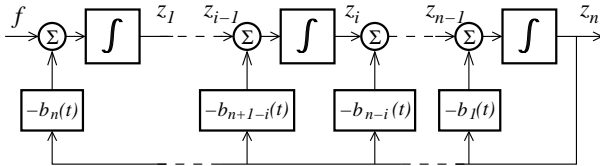


Fig. 4. An second series realization.

The set of associated equations are obtained as

$$\dot{z}_1 = -b_n z_n + f, \quad (20)$$

$$\dot{z}_i = z_{i-1} - b_{n+1-i} z_n \quad (i = 2, \dots, n). \quad (21)$$

Equation (21) yields after  $(i - 1)$  differentiations

$$D^i z_i = D^{i-1} z_{i-1} - D^{i-1} (b_{n+1-i} z_n) \quad (i = 2, \dots, n). \quad (22)$$

Adding all equations in (22), we obtain

$$D^n z_n + \sum_{l=1}^n D^{n-l} (b_l z_n) = f. \quad (23)$$

Since the Leibniz-rule of differentiation gives

$$D^{n-l} (b_l z_n) = \sum_{k=0}^{n-l} \binom{n-l}{k} D^{n-l-k} b_l D^k z_n, \quad (24)$$

equation (23) can be rewritten as

$$D^n z_n + \sum_{k=0}^{n-1} \left[ \sum_{l=1}^{n-k} \binom{n-l}{k} D^{n-l-k} b_l \right] D^k z_n = f. \quad (25)$$

Next, we conclude that if

$$a_{n-l} = \sum_{l=1}^{n-k} \binom{n-l}{k} D^{n-l-k} b_l, \quad (26)$$

while

$$z_n = x, \quad (27)$$

then (25) is equivalent to (1).

Now, it is observed that for LTI systems equation (26) reduces to

$$a_{n-k} = b_{n-k}. \quad (28)$$

Then, and only then, the realizations in figure 1 and figure 4 are equivalent.

Finally, a small change in the realization scheme of figure 4 is applied, see figure 5 as a result. There we have

$$\left. \begin{aligned} \dot{z}_1 &= b_n z_n - f \\ &\vdots \\ \dot{z}_i &= -z_{i-1} + b_{n+1-i} z_n \\ &\vdots \\ \dot{z}_n &= -z_{n-1} + b_1 z_n \end{aligned} \right\}. \quad (29)$$

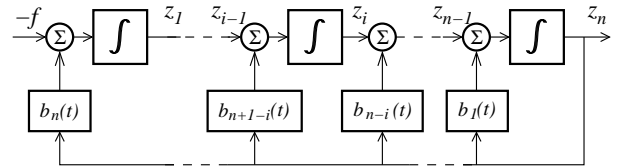


Fig. 5. The series realization represents the adjoint system.

This yields the following collection of equations

$$\left. \begin{aligned} (-1)^n D^n (z_n) + (-1)^{n-1} D^{n-1} (b_1 z_n) &= \\ &(-1)^{n-1} D^{n-1} z_{n-1} \\ &\vdots \\ (-1)^2 D^2 (z_2) + (-1)^1 D^1 (b_{n-1} z_n) &= (-1)^1 D^1 z_1 \\ (-1)^1 D^1 (z_1) + (-1)^0 D^0 (b_n z_n) &= f, \end{aligned} \right\} \quad (30)$$

As a consequence, it follows that

$$\begin{aligned} &(-1)^n D^n (z_n) + (-1)^{n-1} D^{n-1} (b_1 z_n) + \dots + \\ &+ (-1)^1 D^1 (b_{n-1} z_n) + (-1)^0 D^0 (b_n z_n) = f \end{aligned} \quad (31)$$

Thus, the second series scheme of figure 5 represents the adjoint equation of (1) iff

$$b_i = a_i . \quad (32)$$

## V. CONCLUSIONS

A variety of canonical representations for LTV signal processors is derived. They are realized with integrators, time-dependent multipliers and adders, respectively. Most of the proposed processors are generalizations of known topologies for constant systems. It turns out that the classical eigenvalues in the time-invariant context have to be replaced by the earlier introduced dynamic eigenvalues.

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